# POLYNOMIAL BEHAVIOR OF THE HONDA FORMAL GROUP LAW

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Abstract. This note provides the calculation of the formal group law $F(x, y)$ in modulo $p$ Morava $K$-theory at prime $p$ and $s > 1$ as an element in $K(s)^*[x][[y]]$ and some applications to relevant examples.

## 1. Introduction

Let $K(s)^*(-)$, be the $s$-th Morava $K$-theory at prime $p$ [4]. The coefficient ring $K(s)^*(pt)$ is the Laurent polynomial ring in one variable $\mathbb{F}_p[v_s, v_s^{-1}]$, where $\mathbb{F}_p$ is the field of $p$ elements and $deg(v_s) = -2(p^s - 1)$.

Let $F(x, y)$ be the formal group law in $K(s)^*(-)$ theory [5]. The purpose of this note is to prove that if $s > 1$, the formal group law $F(x, y)$ is a polynomial modulo $y^{p^{i(s-1)}}$ (or equivalently modulo $x^{p^{i(s-1)}}$) for any $i \geq 1$, see Theorem 2.1. This fact was never mentioned before in the literature even though the proof is quite simple. We also want to have a method for explicit calculation. The idea is to apply the Ravenel formula [8] involving Witt's symmetric polynomials. The proof does not work for $s = 1$. The particular case (2.3) of Theorem 2.1 was applied in several papers by the author, also [10, 11, 12], and [6]. Some other motivation is given in Section 3.

## 2. The statement

Recall the recursive formula from Ravenel's green book, (see [8], 4.3.8) for the formal group law. In $K(s)^*(-)$ theory it reads (we set $v_s = 1$ and $q = p^{s-1}$ )

$$F(x, y) = F(x + y, w_1(x, y)^q, w_2(x, y)^{q^2}, w_3(x, y)^{q^3}, \cdots) \tag{2.1}$$

where $F(x, y, z, \cdots) = x \oplus_F y \oplus_F z \oplus_F \cdots$ is the iterated $x \oplus_F y = F(x, y)$ and $w_j$ are $\mod (p)$ Witt's integral symmetric homogeneous polynomials of degree $p^j$ :

$$x^{p^n} + y^{p^n} = \sum_j p^j w_j(x, y)^{p^{n-j}}.$$

In particular

$$\begin{aligned} w_0 =& x + y, \\ w_1 =& -\sum_{0<j<p} p^{-1} \binom{p}{j} x^j y^{p-j}. \end{aligned}$$

We will need that $deg(w_j) = p^j$ and that $w_j(x, 0) = w_j(0, y) = 0$, for $j > 0$.

Clearly we have

$$F(x, y) = x + y \text{ modulo } y^q. \tag{2.2}$$

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One has for $s>1$ (see [1])

$$F(x,y)=x+y+w_1(x,y)^q \text{ modulo } y^{q^2}. \tag{2.3}$$

We now want to prove that for $s>1$, $F(x,y)$ is again a polynomial modulo $y^{q^n}$ for any $n$. The idea is to apply (2.1).

**Theorem 2.1.** *One has $F(x,y)\in K(s)^*[x][[y]]$ for the formal group law $F(x,y)$ in mod $p$ Morava $K(s)^*(-)$ theory at $p$ and $s>1$.*

*A method for calculation of $F(x,y)$ modulo $y^{q^n}$ is given by the Ravenel formula* (2.1) *and induction on $n$.*

*Proof.* By induction hypothesis, we have that $F(x,y)$ modulo $y^{q^k}$, $k\le n$ is a polynomial, say $P_k(x,y)$. By (2.2) and (2.3) we have

$$P_1(x,y)=x+y,\ \ P_2(x,y)=x+y+w_1(x,y)^q.$$

Induction step: Let us work modulo $y^{q^{n+1}}$. Then (2.1) implies

$$F(x,y)\equiv F(x+y,w_1(x,y)^q,w_2(x,y)^{q^2},\cdots,w_n(x,y)^{q^n}).$$

By induction hypothesis we have

$$\begin{aligned}
F(x,y)\equiv & z_1+w_n^{q^n}, && \text{where}\quad z_1=F(x+y,w_1^q,\cdots,w_{n-1}^{q^{n-1}});\\
z_1\equiv & P_2(z_2,w_{n-1}^{q^{n-1}}), && z_2=F(x+y,w_1^q,\cdots,w_{n-2}^{q^{n-2}});\\
z_2\equiv & P_3(z_3,w_{n-2}^{q^{n-2}}), && z_3=F(x+y,w_1^q,\cdots,w_{n-3}^{q^{n-3}});\\
\cdots & && \\
z_{n-2}\equiv & P_{n-1}(z_{n-1},w_2^{q^2}), && z_{n-1}=F(x+y,w_1^q);\\
z_{n-1}\equiv & P_n(x+y,w_1^q). &&
\end{aligned}$$

Accordingly $F(x,y)$ is again a polynomial modulo $y^{q^{n+1}}$ for any natural $n$. Therefore one can collect the coefficients at $y^j$, $j<q^{n+1}$ for any $n$ and write

$$F(x,y)=\sum A_l(x)y^l\in K(s)^*[x][[y]].$$

□

The proof above gives more, namely one can evaluate the degree of the polynomial $A_l(x)$ .

**Proposition 2.2.** *In $F(x,y)=\sum \alpha_{ij}x^iy^j$ we have $\alpha_{ij}=0$ for $i>(pq)^n$ whenever $j<q^n$.*

*Proof.* Base case is obvious. Induction step: By Theorem 2.1 we have modulo $y^{q^{n+1}}$

$$F(x,y)\equiv P_n(x+y,z)+w_n(x,y)^{q^n},\quad z=F(w_1(x,y)^q,\cdots,w_{n-1}(x,y)^{q^{n-1}}).$$

The term $w_n^{q^n}$ is of degree $(pq)^n$ hence is irrelevant.

Let $\beta_{ij}(x+y)^iz^j$ be any term of the polynomial $P_n(x+y,z)$. By induction hypothesis we have

$$i\le (pq)^n \text{ whenever } j<q^n.$$

Then $z^j$ is a polynomial in $w_1^q,\cdots,w_{n-1}^{q^{n-1}}$. Therefore it has the terms

$$(w_1^q)^{j_1}\cdots(w_{n-1}^{q^{n-1}})^{j_{n-1}},\ \text{with}\quad j_1<q^n,\cdots j_{n-1}<q^2,$$

as we work modulo $y^{q^{n+1}}$.

Therefore $z^j$, as a polynomial in $x$ and $y$, has the terms of total degree

$$pqj_1 + (pq)^2 j_2 + \cdots + (pq)^{n-1} j_{n-1} < pq^{n+1} + p^2 q^{n+1} + \cdots + p^{n-1} q^{n+1}.$$

Thus for any term of $P_n(x+y, z)$

$$\text{the total degree} < (pq)^n + q^{n+1} \sum_{1 \le l \le n-1} p^l < q^{n+1} \sum_{1 \le l \le n} p^l < (pq)^{n+1}.$$

This completes the proof.

□

## 3. Some simple applications

The particular case $n = 2$ of Theorem 2.1 was already applied in several papers.

Consider an extension of $C_{p^k}$ by an elementary abelian $p$-group. That is $G$ fits into an extension

$$1 \to (C_p)^l \to G \to C_{p^k} \to 1.$$

It is known [13, 7] that $G$ is good, i.e., $K(s)^*(BG)$ is generated by Chern classes. However the explicit account of the ring structure was never done for $k > 1$.

The examples for the case $k = 1$ was considered in [2], [3]. Namely let $\xi$ be a complex $m$-plane bundle over the total space of a cyclic covering $\pi : X \to X/C_p$ of prime index $p$. Let $c$ be the Chern class of $X \times_{C_p} \mathbb{C} \to X/C_p$, the complex line bundle associated to covering $\pi$ . In [1] we showed that modulo image of the transfer homomorphism the $i$-th Chern class $c_i$ of the transferred bundle $\xi$ can be written as a polynomial $\mathcal{A}_i$ in Chern classes $c_p, c_{2p}, \cdots c_{mp}$ and $c^{p-1}$. Using the polynomials $\mathcal{A}_i$ in [2], [3], we showed for various examples of finite groups that $K(s)^*(BG)$ is the quotient of a polynomial ring by an ideal for which we listed explicit generators.

We recall that Morava $K$-theory for a cyclic group is the truncated polynomials [9]. In particular

$$K(s)^*(BC_{p^k}) = \mathbb{F}_p[v_s, v_s^{-1}][u]/u^{p^{ks}}.$$

Also

$$K(s)^*(BU(m)) = \mathbb{F}_p[v_s, v_s^{-1}][[c_1, \ldots, c_m]]$$

and because of the Künneth isomorphisms

$$K(s)^*(BU(m) \times BC_{p^k}) = K(s)^*(BU(m)) \otimes K(s)^*(BC_{p^k}).$$

Theorem 2.1 enables to write explicitly the relations derived by formal group law and splitting principle as relations in Chern classes of complex representations.

In particular, let $\theta$ be the line complex bundle over $BG$, associated to covering $\pi : BH \to BG$, $H = (C_p)^l$, $\eta$ is the pullback by projection on the first factor $H \to C_p$ of the canonical bundle over $BC_p$ and $\pi_! \eta$ is the transferred $\eta$. Then we have the bundle relation over $BG$

$$\pi_! \eta \otimes \theta = \pi_! \eta. \tag{3.1}$$

The relation (3.1) holds because of Frobenius reciprocity of the transfer homomorphism of covering $\pi$ in complex $K$-theory:

$$\pi_! \eta \otimes \theta = \pi_!(\eta \otimes \pi^*(\theta)) = \pi_!(\eta \otimes 1) = \pi_! \eta.$$

This implies the relations

$$c_i(\pi_! \eta \otimes \theta) = c_i(\pi_! \eta)$$

in $K(s)^*(BG)$. If we want to write everything in the explicit form, we have to apply the splitting principle to (3.1) and write formally $\pi_! \eta$ as the sum of line bundles $\pi_! \eta = \eta_1 + \cdots + \eta_{p^k}$. Thus we have

$$\eta_1 + \cdots + \eta_{p^k} = \eta_1 \otimes \theta + \cdots + \eta_{p^k} \otimes \theta.$$

Using the elementary symmetric polynomials $\sigma_i$, $i = 1, \cdots, p^k$ we can write

$$c_i(\pi_! \eta) = \sigma_i(c_1(\eta_1), \cdots, c_1(\eta_{p^k})).$$

In fact we have the following equations

$$\sigma_i(c_1(\eta_1), \cdots, c_1(\eta_{p^k})) = \sigma_i(F(c_1(\eta_1), c_1(\theta)), \cdots, F(c_1(\eta_{p^k}), c_1(\theta))). \tag{3.2}$$

To rewrite (3.2) explicitly, we apply Theorem 2.1 for each term $F(c_1(\eta_j), c_1(\theta))$ and write it as a polynomial in $c_1(\eta_j)$ and $u = c_1(\theta)$ as $\theta^{p^k} = 1$ implies $u^{p^{ks}} = 0$. This is because $c_1(\theta^{p^k}) = [p^k](c_1(\theta))$ and $[p](x) = x^{p^s}$ for the Honda formal group law.

Finally we turn to the Chern classes. In this way one can try to compute $K(s)^*(BG)$ as a quotient of a polynomial ring (as $G$ is finite) by relations ideal. For this we have to establish two facts: the classes we define generate, and the list of relations is complete. To check the latter is easier if the relations are given as explicit polynomials.

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